Question

Which method of stacking cannonballs is a more efficient use of space, with a square base or triangular base?

Helpful equations

Volume of pyramid = base ×height/3

Volume of sphere of radius r = $\frac{4\pi r^3}{3}$

$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum_{x=1}^{n} \frac{x(x+1)}{2} = \frac{n(n+1)(n+2)}{6}$

Answer

They are both the same. As the stacks get bigger, they approach a ratio of efficiency of

$\frac{\pi\sqrt{2}}{6}$ =~ 74.05%.

Solution

Let's start with the tetrahedron.

The Pythagorean formula tells us that for a right triangle of side 1:

Height = $\frac{\sqrt{3}}{2}$

Distance from side to center = $\frac{\sqrt{3}}{6}$

Distance from corner to center = $\frac{\sqrt{3}}{3}$

Area = base*height/2 = $\frac{\sqrt{3}}{2}$

Distance from side to center = $\frac{\sqrt{3}}{4}$

Now consider a tetrahedron of side 1. From a right triangle consisting of the top corner any base corner and the center of the base. The hypotenuse is 1 and the distance from the bottom corner to base is $\frac{\sqrt{3}}{6}$. The Pythagorean formula can give us the height as $\sqrt{1-(\frac{\sqrt{3}}{3})^2}$ = $\frac{\sqrt{6}}{3}$.

The volume of the tetrahedron is base * height / 3.

We know the base and height, so the volume is $\frac{\sqrt{2}}{12} = \sim 0.117851.$

The number of balls in a tetrahedron-shaped pyramid with n balls along one side is:

$\frac{n(n+1)(n+2)}{6}$ = $\frac{n^3+3n^2+2n}{6}$

If each sphere has a radius of 1, then the volume of each sphere is $\frac{4\pi}{3}$.

So, a tetrahedron-shaped pyramid of sphere of volume 1 and side length n will have a total volume of the sphere is $2\pi\frac{n^3+3n^2+2n}{9}$.

The size of a tetrahedron enclosing this stack would need a side length of 2n because the diameter of each sphere is 2.

The total volume of this tetrahedron will be $8n^3 \times \frac{\sqrt{2}}{12} = \frac{2\sqrt{2}n^3}{3}$

The efficiency ratio of the area of the spheres to area of the tetrahedron is:

$$\frac{2\pi\frac{n^3 + 3n^2 + 2n}{9}}{\frac{2\sqrt{2}n^3}{3}}$$

As n approaches infinity, the n and n^2 terms will become negligible. So we can simplify the ratio to:

$$\frac{2\pi\frac{n^3}{9}}{\frac{2\sqrt{2}n^3}{3}}$$

Let's divide n^3 from both sides:

$\frac{\frac{2\pi}{9}}{\frac{2\sqrt{2}}{3}} = \frac{\sqrt{2}\pi}{6}$ =~ 74.05%

Next, let's consider the pyramid with a square base.

You can see from the picture the length of a side of the base equals the distance from a corner of the base to the top of the pyramid. This is because they both the sum of the diameters of the number of balls along the side of the base.

The Pythagorean formula tells us that the height of the pyramid of side length 1 is $\sqrt{2}/2$. Again, the area of a pyramid is base*height/3. In the case of a square-based pyramid of side length 1, the area is $\sqrt{2}/6$. If we multiply the base by 2n, we multiply the area by the cube of that, or $8n^3$. So, for a side of length 2n, the area is $\frac{4\sqrt{2}n^3}{3}$.

If the number of spheres along the side is n, then the total number of spheres will be:

$\frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+3n}{6}$.

With an area of each sphere of $\frac{4\pi}{3}$, the total volume of the spheres will be $\pi\,\frac{8n^3+12n^2+12n}{18}$ =

$\pi\,\frac{4n^3+6n^2+6n}{9}$

The ratio of efficiency will be:

$$\frac{\pi\,\frac{4n^3+6n^2+6n}{9}}{\frac{4\sqrt{2}n^3}{3}}$$

As n approaches infinity, the n and n^2 terms will become insignificant, allow us to simplify the equation to:

$$\frac{\pi\,\frac{4n^3}{9}}{\frac{4\sqrt{2}n^3}{3}}$$

Let's divide both the numerator and denominator by n^3 giving us an efficiency ratio of:

$\frac{\pi\frac{4}{9}}{\frac{4\sqrt{2}}{3}} = \frac{\sqrt{2}\pi}{6}$ =~ 74.05%

So, both methods are equally efficient!